MATH-305
Numerical Methods

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3. Bisection Mthd

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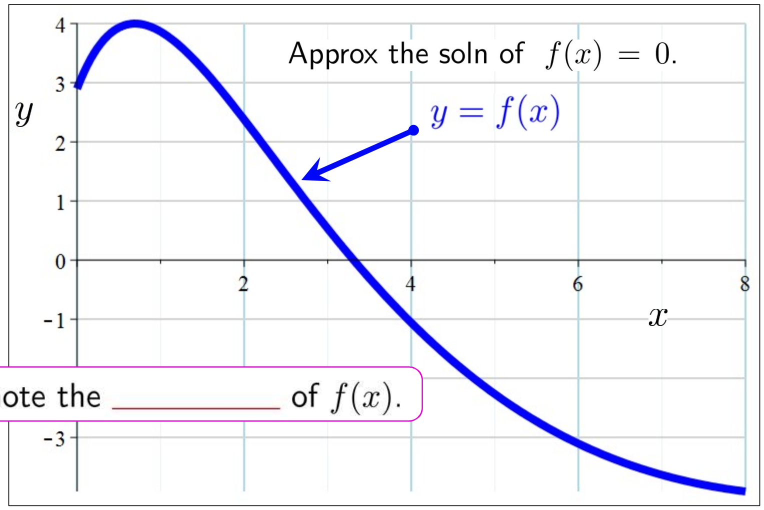
Ch. 1 Zeros of Functions

Here we seek a soln of a single ______________ eqn involving one unknown.

Example 1: Solve the eqn for x: $e^{-x} = \cos x$.

Zero Form	Root Form
$e^{-x} - \cos x = 0$	$e^{-x} = \cos x$
$f(x) = 0$	Any eqn _____ in zero form.
A soln is called a ______ of f.	A soln is called a _____ of eqn.
A zero of f is an _____________ of the _____________ of f.	Most of the mthds require the eqn in _________ **form**.

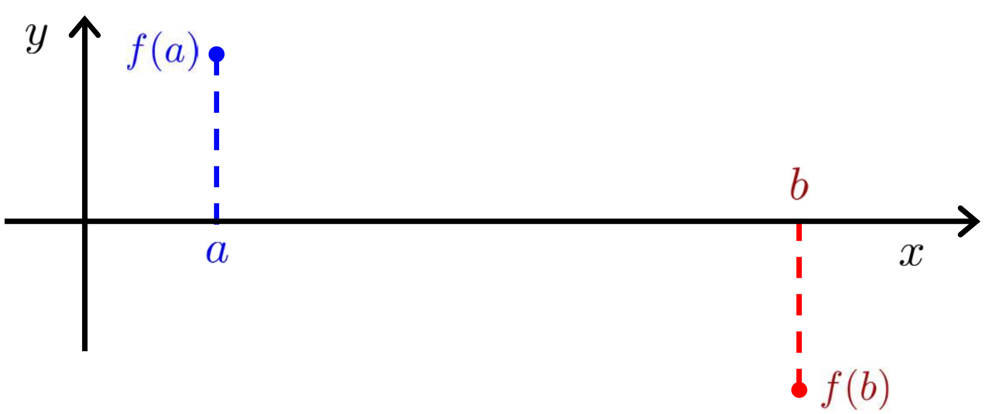
Zeros of Functions



We'll let z denote the __________ of $f(x)$.

Intermediate Value Theorem (IVT)

Supp. ftn $f(x)$ is ________________ on interval $[a, b]$ and that $f(a)$ and $f(b)$ have _____ signs. Then $f(x)$ has ____________ one zero b/w ______________.



Bisection Method | Interval Halving

To approx. a zero of ftn $f(x)$, select a starting interval $[a, b]$ satisfying the ______. We reserve a and b for the ___________________________ of the ___________________________ interval.

For computational convenience, we denote a by x_ℓ and b by x_r, and let

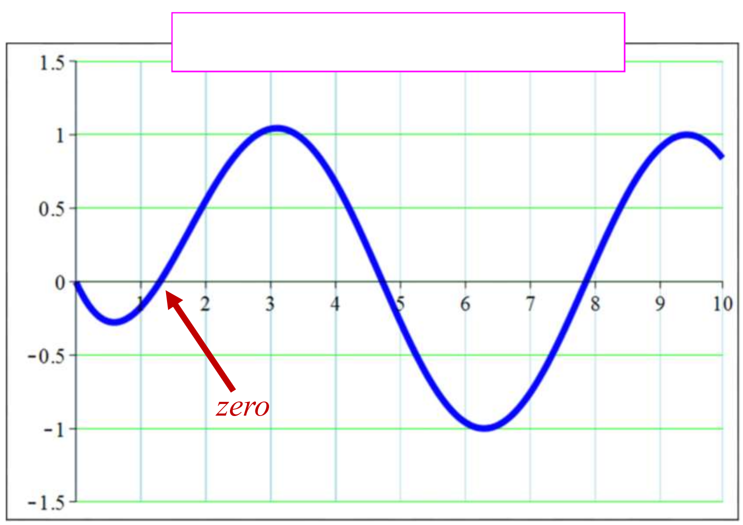
Ensure that y_ℓ and y_r have ___________________________ signs.*

* So the points $(x_\ell\,, y_\ell)$ and $(x_r\,, y_r)$ are on the ______________ of ftn $f(x)$ and are on opposite sides of the ___ axis.

Solve $f(x) = 0$ for x

Example 2. Approx. the first positive soln of $e^{-x} = \cos x$.

First write the eqn in _____________ form:



Q. What is a good starting interval?

We'll use a starting interval of

$$f(x) = e^{-x} - \cos x$$

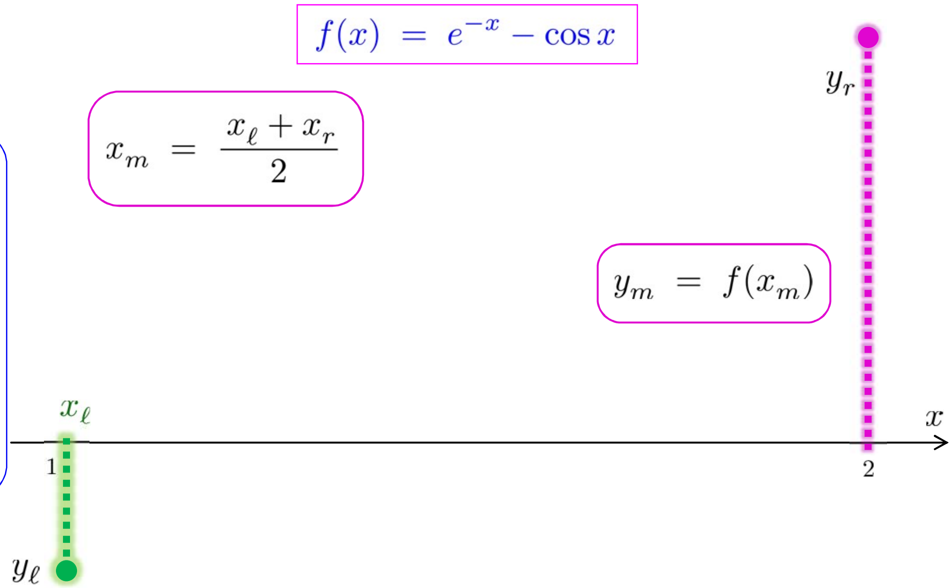
$$x_m^{(1)} =$$

$$x_m^{(2)} =$$

$$x_m^{(3)} =$$

$$x_m = \frac{x_\ell + x_r}{2}$$

$$y_m = f(x_m)$$



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General Definitions

Solve $f(x) = 0$ for x

1. Each x_m, called an ____________________ , approxs the zero z of ftn f.
2. The process of repeating steps to obtain each iterate is called an

   ________________________.

3. Performing iterations is called ________________________.

With enough iterations, the iterates converge to z:

Q. What is

When convenient, we'll let $x_m^{(n)}$ denote the ______ iterate.

E.g., $x_m^{(4)}$ is the ______ iterate.

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Do Ex 2. $f(x) = e^{-x} - \cos x, \quad x_m = \dfrac{x_\ell + x_r}{2}$ Use radians!

Iter $n = 1$: I always use more digits than I display.

$$x_\ell = 1.0\,, \quad y_\ell = f(x_\ell) = f(1) = -0.1724\ldots,$$
$$x_r = 2.0\,, \quad y_r = f(x_r) = f(2) = +0.5514\ldots,$$

From the previous slide.

$$x_m = \frac{x_\ell + x_r}{2} = \qquad , \quad y_m = f(x_m) = +0.1523\ldots$$

Sign Test: $y_\ell \quad y_m \quad y_r$

So the zero of f is b/w

I.e., the zero of f is on interval

$x_m = \dfrac{x_\ell + x_r}{2} \quad f(x) = e^{-x} - \cos x$ Use radians!

Iter $n = 2$:

$$x_\ell = 1.0\,, \quad y_\ell = f(x_\ell) = f(1) = -0.1724\ldots,$$
$$x_r = 1.5\,, \quad y_r = f(x_r) = f(1.5) = +0.1523\ldots,$$

From the previous slide.

$$x_m = \frac{x_\ell + x_r}{2} = \qquad , \quad y_m = f(x_m) = -0.02881\ldots$$

Sign Test: $y_\ell \quad y_m \quad y_r$

So the zero of f is b/w

I.e., the zero of f is on interval

$$x_m = \frac{x_\ell + x_r}{2} \qquad f(x) = e^{-x} - \cos x$$ Use radians!

Iter $n = 3$:

$x_\ell = 1.25, \quad y_\ell = f(x_\ell) = f(1.25) = -0.02881\ldots,$
$x_r = 1.5, \quad y_r = f(x_r) = f(1.5) = +0.1523\ldots,$
From the previous slide.

$x_m = \dfrac{x_\ell + x_r}{2} = \qquad , \quad y_m = f(x_m) = +0.05829\ldots$

Sign Test: $y_\ell \quad y_m \quad y_r$

So the zero of f is b/w

I.e., the zero of f is on interval

Use radians!

$$x_m = \frac{x_\ell + x_r}{2} \qquad f(x) = e^{-x} - \cos x$$

$y_\ell = f(1) = -0.1724\ 2286\ldots$
$y_r = f(2) = +0.5514\ 8211\ldots$

n	x_ℓ	x_m	x_r	y_ℓ	y_m	y_r	$\lvert x_r - x_\ell\rvert$
1	1.0000 0000	1.5000 0000 R	2.0000 0000	−	+0.152392 R	+	0.5000 0000
2	1.0000 0000	1.2500 0000 L	1.5000 0000	−	−0.028817 L	+	0.2500 0000
3	1.2500 0000	1.3750 0000 R	1.5000 0000	−	+0.058291 R	+	0.1250 0000
4	1.2500 0000	1.3125 0000 R	1.3750 0000	−	+0.013712 R	+	0.0625 0000
5	1.2500 0000	1.2812 5000 L	1.3125 0000	−	−0.007827 L	+	0.0312 5000
6	1.2812 5000	1.2968 7500 R	1.3125 0000	−	+0.002876 R	+	0.0156 2500

Solve $f(x) = 0$ for x

When do we stop iterating? **As we iterate**, we should see that

a) the ____________ approach _________:

b) the difference b/w ______________________________ approaches ____:

c) the _____________ of the interval containing zero z decreases to ____:

That means that each of these 3 quantities becomes _____________________.

Therefore...

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Solve $f(x) = 0$ for x

Possible Stopping Tests

Let TOL, called a __________________ , denote some __________ positive no. of our choosing. We stop iterating and accept the last x_m as our approx. of zero z when:

a)

b)

c) **IST**

The latter is called the _________________________________ (**our favorite**).

Remember the importance of understanding ______________!

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Solve $f(x) = 0$ for x

Observations:

1. At each iter'n, the ___________ of interval $[x_\ell, x_r]$ is _______________ .
2. Since both x_m and z are on that interval, their difference _____________ _______________ the interval's length:
3. Consequently, the interval length is ______________________________:

In our ex, the interval length after 6 iter's is ________________ , so

$\left(\text{Actual error:} \quad |\,\text{Error}_6\,| = 0.004179\ldots\right)$

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Solve $f(x) = 0$ for x

Error Bound in using x_m to approx zero z:

(EB.1)

So if we use the **interval stopping test** — *stop iterating when*

IST

then our tolerance TOL is an __________________________________:

(EB.2)

So unlike the other stopping tests, with the IST our tolerance TOL ______________ represents the _______________________________ we will __________________!

Furthermore:

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Let $L = b - a$ denote the length of the starting interval $[a, b]$.

Then the error:

1. after 1 bisection is

$$|\,\text{Error}_1\,| \;\leq$$

2. after 2 bisections is

$$|\,\text{Error}_2\,| \;\leq$$

3. after 3 bisections is

$$|\,\text{Error}_3\,| \;\leq$$

n. after n bisections is

$$|\,\text{Error}_n\,| \;\leq$$

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Solve $f(x) = 0$ for x

So the error bound after n bisections is

$$|\,\text{Error}_n\,| \;\leq \qquad \text{(EB.3)}$$

I.e., the nth iterate $x_m^{(n)}$ must be within __________ units of the zero z of f .

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Solve $f(x) = 0$ for x

$$|\,\text{Error}_n\,| \;\leq\; \frac{b-a}{2^n} \qquad \text{(EB.3)}$$

Example 3.

Supp. ftn $f(x)$ satisfies the IVT on interval $[-2, 6]$. Determine the error bound if we bisect ______ times.

$$|\,\text{Error}_7\,| \;\leq$$

So the 7th iterate $x_m^{(7)}$ must be within ______________ units of zero z .

If we can tolerate that much error, then we bisect 7 times and accept $x_m^{(7)}$.

Example 4.

$$|\,\text{Error}_n\,| \;\leq\; \frac{b-a}{2^n}$$

Supp. ftn $f(x)$ satisfies the IVT on interval $[-2, 6]$. How many times should we bisect to ensure the error ____________________________ 10^{-3} ?

ANS. We set the error bound to ______ and _______________ for _____.

$$\frac{b-a}{2^n} \overset{\text{set}}{=} 10^{-3} \implies$$

$$\implies$$

$$\implies$$

↑ **Error Bound**

Solve $f(x) = 0$ for x

$\Longrightarrow$

$\Longrightarrow$

$\Longrightarrow \quad n =$

So bisecting ____ **times** ensures that the error will not exceed 10^{-3} units.

Standing Assignment

Always rework ALL of my exs successfully yourself *before* attempting the HW!

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Solve $f(x) = 0$ for x

Mathematical Aside:

Real numbers u and v agree to ____________________ d decimal places if

So if we choose TOL of the form TOL $= 0.5 \times 10^{-d}$ and stop iterating when

then x_m agrees with z to ____________________________ decimal places.

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Solve $f(x) = 0$ for x

Example 5.

Using the starting interval $[-2, 6]$, how many times must we bisect to guarantee ____________________ decimal place agreement b/w x_m and z?

$$|x_r - x_\ell| \overset{\text{set}}{=}$$

$$\Longrightarrow \frac{b-a}{2^n} =$$

$$\Longrightarrow \frac{6-(-2)}{2^n} =$$

$$\Longrightarrow 2^n = \frac{8}{0.5 \times 10^{-4}}$$

$$\Longrightarrow$$

So bisecting ____ **times** ensures that x_m agrees with z to __________ dec. places.

You should rework this ex!

Solve $f(x) = 0$ for x

Properties of Bisection Mthd

1. It is an ___________ **mthd**. At each iter'n, we possess an interval $[x_\ell, x_r]$ that contains zero z.
2. It gives ___-sided convergence toward zero z:
3. B/c of Prop 2, the ____________________ shrinks to ___ , allowing us to use the IST: Stop iterating when

IST

Solve $f(x) = 0$ for x

4. B/c of the IST, our tolerance TOL truly represents an error ____________:

(EB.2)

5. B/c of the IST, choosing TOL of the form 0.5×10^{-d} ensures that x_m agrees with zero z to ___________________________ dec. places.

6. Converges __________________ than other mthds as we'll see.

7. Gives a linear rate of convergence to z:

$$\text{Digits} \approx kn.$$

Solve $f(x) = 0$ for x

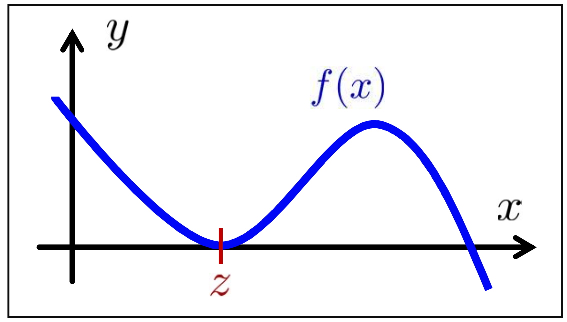
Dec. Place Conv.	Iterations
1	
2	
3	
4	
5	
6	

This mthd gives a ____________________ **rate of convergence**:

accuracy is approx. proportional to n $\Longrightarrow$

READ. **Try to answer these questions *before* reading the answers!**

1. Explain why Bisection mthd cannot be used to approx. the zero shown, where ftn f touches the x axis tangentially at zero z without crossing through the x axis.
2. How would you approach this problem?



Answers:

1. If one selects a small starting interval $[a,\, b]$ containing zero z, then $f(a)$ and $f(b)$ will **not** have opposite signs, so the conditions of the Intermediate Value Thm will not be satisfied.
2. Since f touches the x axis tangentially at $x = z$, then $x = z$ is a zero of both $f(x)$ **and** $f'(x)$. Furthermore, $f'(x)$ changes sign across $x = z$. So we may apply Bisection to ftn $f'(x)$.

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NOTE:

It might be tempting to select our tolerance TOL to be *very* small, such as 10^{-15}, for example. This can be problematic! Recall the number **machine epsilon** from previous notes. It is possible that we select TOL to be **so small** that the the stopping test

$$|\, x_r - x_\ell \,| \;\leq\; \text{TOL}$$

is **never** satisfied, in which case the code might never stop iterating, even though our iterates $x_m^{(n)}$ become *extremely* close to zero z of $f(x)$.

Recall that you are to **study all** pages of the notes, including those pages not covered in class. They are usually marked with 📝.

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Bisection Method Algorithm

With (x_ℓ, y_ℓ) and (x_r, y_r) in hand:

1. Compute iterate x_m.
2. Compute $y_m = f(x_m)$.
3. Apply the Sign Test (IVT):

   If y_ℓ and y_m have opp. signs,

   store x_m in x_r and y_m and y_r.

   Otherwise

   store x_m in x_ℓ and y_m and y_ℓ.
4. Print: iteration number, x_m, and $f(x_m)$.

Repeat Steps $1 - 4$ until the IST is satisfied.

Recall that an algorithm is **NOT** a code.

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